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Periodic Even Degree Spline Interpolation on a Uniform Partition

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Periodic even degree spline interpolants of a function f at the knots are considered. Existence and uniqueness results are proved, and error bounds of the form $\|f^{(k)}-s^{(k)}\|_\infty \leqslant \sigma_{r,k} h^{2r+1-k}\{\|f^{(2r+1)}\|_\infty + \mathrm{Var}(f^{(2r+1)})\}$ $(k=0,..., 2r)$ are obtained.

1. INTRODUCTION

Let $\Delta=\{x_i\}_{i=0}^N$ be a *uniform partition* of $[a, b]$, $a=x_0<\cdots<x_N=b$, and $x_i=a+ih$, where $h=(b-a)/N$. An *even degree spline* is a function $s\in C^{2r-1}[a, b]$ such that s restricted to $[x_i, x_{i+1}]$ is a polynomial of degree at most $2r$. It is a *periodic even degree spline* if $s^{(k)}(a)=s^{(k)}(b)$ $(k=0,..., 2r-1)$. In this paper we define a periodic even degree spline from its nodal values $s(x_i)$ $(i=0,..., N)$.

Nice error bounds have been established for periodic odd degree spline interpolation (e.g., see Ahlberg et al. [2], B. Swartz [11, 12], Albasiny and Hoskins [3], and recently, T. R. Lucas [10]). It appears that we can obtain similar results for periodic even degree spline interpolation. In [7], extending the results of Daniel [5] and de Boor [4], we have studied periodic quadratic spline interpolation and showed that good results are obtained when the partition is uniform. In [8] we have obtained similar results for periodic quartic spline interpolation on a uniform partition. The object of this paper is the study of periodic even degree spline interpolation on a uniform partition. We show existence and uniqueness of periodic even degree spline interpolants and obtain error bounds of the form

$$\|f^{(k)}-s^{(k)}\|_\infty \simeq O(h^{2r+1-k}) \qquad (k=0,..., 2r).$$

These results are also extensions of those obtained by Meek [13].

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Throughout this paper we will use the following notations. If $g: [a, b] \to R$ is a given function, we will write $g_i = g(x_i)$, $x_{i+(1/2)} = (x_i + x_{i+1})/2$, $g_{i+(1/2)} = g(x_{i+(1/2)})$, $g^{(k)}$ is the kth derivative of g, $g^{(0)} = g$ and $\mathrm{Var}(g)$ is the total variation of g on $[a, b]$. We also consider the following function spaces: $C^k[a, b]$, the spaces of functions with continuous derivatives through order k, $C_p^k[a, b]$, the spaces of functions $f \in C^k[a, b]$, such that $f^{(l)}(a) = f^{(l)}(b)$ for all $l = 0, \ldots, k$, and $\mathscr{P}_k$, the space of all polynomials of degree at most k.

2. Existence of Even Degree Periodic Splines

As previously defined, on each interval $[x_i, x_{i+1}]$ a periodic spline of degree $2r$ can be written

$$s(x) = \sum_{k=0}^{2r} s_i^{(k)} \frac{(x - x_i)^k}{k!} \tag{1}$$

and our first step is to relate the quantities $s_i^{(k)}$ $(k = 1, \ldots, 2r)$ to the quantities s_i. This is done by the following fundamental relationship, proved by Fyfe [9, Theorem 1],

$$\sum_{j=0}^{2r-1} C_{j,2r}^{(0)} s_{i+j}^{(k)} = \frac{(2r)_k}{h^k} \sum_{j=0}^{2r-1} C_{j,2r}^{(k)} s_{i+j} \tag{2}$$

for all $k = 1, \ldots, 2r-1$, where $(2r)_k = (2r)!/(2r-k)!$,

$$C_{j,2r}^{(k)} = \nabla^{2r+1} n_+^{2r-k} \qquad \text{when} \quad n = 2r - j$$

for $k = 0, \ldots, 2r-1$, ∇ is the backward difference operator and $z_+ = (z + |z|)/2$. Finally, for $k = 2r$ we obtain directly from (1)

$$s_i^{(2r)} = \frac{s_{i+1}^{(2r-1)} - s_i^{(2r-1)}}{h} \tag{3}$$

If we consider (2) for $i = -r+1, \ldots, N-r$, and remember that indices must be considered modulo N, we obtain the linear systems

$$C_{2r}^{(0)} s_\Delta^{(k)} = \frac{(2r)_k}{h^k} C_{2r}^{(k)} s_\Delta \tag{4}$$

for all $k = 1, \ldots, 2r-1$, where $s_\Delta^{(k)} = (s_0^{(k)}, s_1^{(k)}, \ldots, s_{N-1}^{(k)})$ and $C_{2r}^{(k)}$ is a band circulant matrix with nonzero elements in a general row consisting of

$$C_{0,2r}^{(k)}, C_{1,2r}^{(k)}, \ldots, C_{r-1,2r}^{(k)}, \ldots, C_{2r-2,2r}^{(k)}, C_{2r-1,2r}^{(k)}$$

with the element $C_{r-1,2r}^{(k)}$ on the diagonal.

Using the properties of the coefficients of the matrix $C_{2r}^{(0)}$ (see the Appendix), we can prove the following result.

THEOREM 1. *Let $\Delta = \{x_i\}_{i=0}^{N}$ be a uniform partition of $[a, b]$. A periodic spline of degree $2r$ is uniquely determined by its nodal values $\{s_i\}_{i=0}^{N-1}$ if and only if N is odd. In this case*

$$s_\Delta^{(k)} = \frac{(2r)_k}{h^k} (EC_{2r}^{(0)})^{-1}\, EC_{2r}^{(k)} s_\Delta$$

where the matrix E of order N is

$$E = \frac{1}{2} \begin{bmatrix} 1 & -1 & 1 & \cdots & -1 & 1 \\ 1 & \ddots & \ddots & \ddots & & -1 \\ -1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & & \ddots & \ddots & \ddots & -1 \\ -1 & 1 & \cdots & -1 & 1 & 1 \end{bmatrix} \tag{5}$$

and $EC_{2r}^{(0)}$ is a symmetric band circulant matrix of order N such that

$$\|(EC_{2r}^{(0)})^{-1}\|_\infty \leqslant \frac{2r+2}{2^{2r+1}(2^{2r+2}-1)} \cdot \frac{(-1)^r}{B_{2r+2}} \tag{6}$$

where B_{2r+2} is a Bernoulli number (see Abramowitz and Stegun [1]). If N is even, the spline does not exist or is not uniquely determined.

Proof. See Dubeau [6]. Q.E.D.

3. DERIVATION OF ERROR BOUNDS

Given a function $f \in C_p^{2r+1}[a, b]$ and a uniform partition $\Delta = \{x_i\}_{i=0}^{N}$, N *odd*, of the interval $[a, b]$. We consider the periodic spline interpolant s of degree $2r$ of f such that $s(x_i) = f(x_i)$. On each interval $[x_i, x_{i+1}]$ the kth derivative $(0 \leqslant k \leqslant 2r)$ of the *remainder function* $e(x) = f(x) - s(x)$ can be written

$$e^{(k)}(x) = \sum_{l=k}^{2r} e_i^{(l)} \frac{(x-x_i)^{l-k}}{(l-k)!} + R_{2r-k}(f^{(2r+1)}; x_i)(x) \tag{7}$$

where

$$R_m(g;\alpha)(x)=\int_\alpha^x \frac{(x-\xi)^m}{m!}\,g(\xi)\,d\xi.$$

The problem is then reduced to the study of the terms $e_i^{(k)}$.

Let us introduce the following compact notation

$$\delta_i^k g=\sum_{j=0}^{2r-1} C_{j,2r}^{(k)}\,g_{i+j}.$$

Hence (2) becomes

$$\delta_i^0 s^{(k)}=\frac{(2r)_k}{h^k}\,\delta_i^k s$$

for all $k=1,\ldots,2r-1$. So, for the error function we obtain

$$\delta_i^0 e^{(k)}=\delta_i^0 f^{(k)}-\frac{(2r)_k}{h^k}\,\delta_i^k f. \tag{8}$$

We are now able to prove the following lemma.

LEMMA 1. *Let N be an odd integer, $\Delta=\{x_i\}_{i=0}^N$ be a uniform partition of $[a,b]$ and $f\in C_p^{2r+1}[a,b]$. Then*

$$\delta_i^0 e^{(k)}=\delta_i^0 R_{2r-k}(f^{(2r+1)};x_{i+r-(1/2)})-\frac{(2r)_k}{h^k}\,\delta_i^k R_{2r}(f^{(2r+1)};x_{i+r-(1/2)}) \tag{9}$$

for all $k=1,\ldots,2r-1$.

Proof. Consider the following Taylor expansions

$$f^{(k)}(x)=p^{(k)}(x)+R_{2r-k}(f^{(2r+1)};x_{i+r-(1/2)})(x) \tag{10}$$

for all $k=0,\ldots,2r-1$, where

$$p^{(k)}(x)=\sum_{l=k}^{2r} f_{i+r-(1/2)}^{(l)}\,\frac{(x-x_{i+r-(1/2)})^{l-k}}{(l-k!)}.$$

Using the notation (A.1) of the Appendix, we have

$$\delta_i^0 p^{(k)}=\sum_{l=k}^{2r} f_{i+r-(1/2)}^{(l)}\,h^{l-k}\gamma_{l-k,2r}^{(0)}$$

and

$$\delta_i^k p = \sum_{l=0}^{2r} f_{i+r-(1/2)}^{(l)} h^l \gamma_{l,2r}^{(k)}.$$

But from (A.2) and (A.3)

$$\delta_i^0 p^{(k)} = \frac{(2r)_k}{h^k} \delta_i^k p \tag{11}$$

and the result follows from (8), (10) and (11). Q.E.D.

In the next two theorems, we obtain bounds for the expressions $e_i^{(k)} + e_{i+1}^{(k)}$ and $e_i^{(k)} - e_{i+1}^{(k)}$. These bounds show us an interesting decomposition of the bound for $e_i^{(k)}$.

THEOREM 2. *Let N be an odd integer, $\Delta = \{x_i\}_{i=0}^N$ be a uniform partition of $[a, b]$ and $f \in C_p^{2r+1}[a, b]$. Then there exist constants $\beta_{r,k}$, independent of the partition, such that*

$$|e_i^{(k)} + e_{i+1}^{(k)}| \leqslant \beta_{r,k} h^{2r+1-k} \, \|f^{(2r+1)}\|_\infty \tag{12}$$

for all $i = 0, \ldots, N-1$ and $k = 1, \ldots, 2r-1$.

Proof. If we write (9) as a linear system

$$C_{2r}^{(0)} e_\Delta^{(k)} = b_{r,k} \tag{13}$$

where $e_\Delta^{(k)} = (e_0^{(k)}, e_1^{(k)}, \ldots, e_{N-1}^{(k)})$ and $b_{r,k}$ is a N-vector whose components are given by

$$(b_{r,k})_i = \delta_{i-r+1}^0 e^{(k)}, \tag{14}$$

and if we use the right-hand side of (9), we have

$$(b_{r,k})_i = \delta_{i-r+1}^0 R_{2r-k}(f^{(2r+1)}; x_{i+(1/2)}) - \frac{(2r)_k}{h^k} \delta_{i-r+1}^k R_{2r}(f^{(2r+1)}; x_{i+(1/2)}) \tag{15}$$

for all $i = 0, 1, \ldots, N-1$. So there exist constants $\alpha_{r,k}$, independent of Δ, such that

$$\|b_{r,k}\|_\infty \leqslant \alpha_{r,k} h^{2r+1-k} \, \|f^{(2r+1)}\|_\infty$$

Now, since $I=E(I+P)$ and $C_{2r}^{(0)}=EC_{2r}^{(0)}(I+P)$, where the matrix E is given by (5) and P is the permutation matrix of order N

$$P=\begin{bmatrix} 0 & 1 & & \bigcirc \\ & \ddots & \ddots & \\ & \bigcirc & \ddots & 1 \\ 1 & & & 0 \end{bmatrix},$$

we obtain

$$(I+P)\,e_{\Delta}^{(k)}=(EC_{2r}^{(0)})^{-1}\,b_{r,k}$$

and the result follows if we set $\beta_{r,k}=\alpha_{r,k}\,\|(EC_{2r}^{(0)})^{-1}\|_{\infty}$. Q.E.D.

THEOREM 3. *Let N be an odd integer, $\Delta=\{x_i\}_{i=0}^{N}$ be a uniform partition of $[a,b]$, $f\in C_p^{2r+1}[a,b]$ with $f^{(2r+1)}$ of bounded variation. Then there exist constants $\beta_{r,k}$, independent of the partition, such that*

$$|e_i^{(k)}-e_{i+1}^{(k)}|\leqslant\beta_{r,k}h^{2r+1-k}\operatorname{Var}(f^{(2r+1)}) \tag{16}$$

for all $i=0,\dots,N-1$ and $k=1,\dots,2r-1$.

Proof. From the system (13) and the relation $(I-P)\,EC_{2r}^{0}=EC_{2r}^{0}(I-P)$, it follows that

$$(I-P)\,e_{\Delta}^{(k)}=(EC_{2r}^{0})^{-1}(I-P)\,Eb_{r,k}. \tag{17}$$

But

$$(I-P)E=\begin{bmatrix} 0 & -1 & 1 & \cdots & -1 & 1 \\ 1 & \ddots & \ddots & \ddots & & -1 \\ -1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & & \ddots & \ddots & \ddots & -1 \\ -1 & 1 & \cdots & -1 & 1 & 0 \end{bmatrix}$$

and (14) imply

$$[(I-P)\,Eb_{r,k}]_i=\sum_{j=i+1}^{i+N-1}(-1)^{j-i}\,\delta_{j-r+1}^{0}\,e^{(k)} \tag{18}$$

for all $i=0,\dots,N-1$.

Using the fact that the partition is uniform and introducing the translation operator $(T_j g)(x) = g(x+jh)$, we can easily show

$$\delta^k_{j-r+1} R_m(g; x_{j+(1/2)}) = \delta^k_{i-r+1} R_m(T_{j-i}\, g; x_{i+(1/2)}) \tag{19}$$

for all $k = 0,\dots, 2r-1$.

Hence, setting $g = f^{(2r+1)}$ in (19) and using (15) and (18), we obtain

$$[(I-P)\, Eb_{r,k}]_i = \delta^0_{i-r+1} R_{2r-k}(\psi; x_{i+(1/2)}) - \frac{(2r)_k}{h^k} \delta^k_{i-r+1} R_{2r}(\psi; x_{i+(1/2)})$$

where

$$\psi = \sum_{j=i+1}^{i+N-1} (-1)^{j-1}\, T_{j-i} f^{(2r+1)}.$$

But

$$|\psi(x)| \leqslant \mathrm{Var}(f^{(2r+1)})$$

so there exist constants $\alpha_{r,k}$, independent of Δ, such that

$$\|(I-P)\, Eb_{r,k}\|_\infty \leqslant \alpha_{r,k} h^{2r+1-k}\, \mathrm{Var}(f^{(2r+1)}) \tag{20}$$

and the result follows from (17) and (20). Q.E.D.

It is now easy to obtain a global error bound for the remainder function.

THEOREM 4. *Let N be an odd integer, $\Delta = \{x_i\}_{i=0}^N$ be a uniform partition of $[a, b]$, $f \in C_p^{2r+1}[a, b]$ with $f^{(2r+1)}$ of bounded variation. Then there exist constants $\sigma_{r,k}$, independent of the partition, such that*

$$\|e^{(k)}\|_\infty \leqslant \sigma_{r,k} h^{2r+1-k} \{\|f^{(2r+1)}\|_\infty + \mathrm{Var}(f^{(2r+1)})\} \tag{21}$$

for all $k = 0,\dots, 2r$.

Proof. Inequalities (12) and (16) imply

$$\|e_\Delta^{(k)}\|_\infty \leqslant \frac{\beta_{r,k}}{2} h^{2r+1-k} \{\|f^{(2r+1)}\|_\infty + \mathrm{Var}(f^{(2r+1)})\}. \tag{22}$$

for $k = 1,\dots, 2r-1$. Moreover, from (3) it follows that

$$e_i^{(2r)} = \frac{e_{i+1}^{(2r-1)} - e_i^{(2r-1)}}{h} - \frac{1}{h} \int_{x_i}^{x_{i+1}} (x_{i+1} - \xi)\, f^{(2r+1)}(\xi)\, d\xi$$

and using (16) we obtain

$$|e_i^{(2r)}| \leqslant \beta_{r,2r-1} h \operatorname{Var}(f^{(2r+1)}) + \frac{h}{2} \|f^{(2r+1)}\|_\infty. \tag{23}$$

The result then follows from (7), (22) and (23). Q.E.D.

Remark 1. From (9) it follows that the right-hand side of (8) is a linear functional L_k which vanishes for all $p \in \mathscr{P}_{2r}$. Thus we can use the Peano kernel theorem (see Davis [14]) to obtain

$$\delta_i^0 e^{(k)} = \int_a^b K_k(t) f^{(2r+1)}(t)\, dt$$

where

$$K_k(t) = \frac{1}{(2r)!} L_{k,x}[(x-t)_+^{2r}].$$

The notation $L_{k,x}[(x-t)_+^{2r}]$ means that the functional L_k is applied to $(x-t)_+^{2r}$ considered as a function of x. Using the following change of variable, $t = x_i + \theta h$, we get

$$\delta_i^0 e^{(k)} = h^{2r+1-k} \int_0^{2r-1} \bar{K}_k(\theta) f^{(2r+1)}(x_i + \theta h)\, d\theta$$

where

$$\bar{K}_k(\theta) = \frac{1}{(2r-k)!} \left[\sum_{j=0}^{2r-1} C_{j,2r}^{(0)} (j-\theta)_+^{2r-k} - \sum_{j=0}^{2r-1} C_{j,2r}^{(k)} (j-\theta)_+^{2r} \right].$$

So, the constants $\alpha_{r,k}$ can be evaluated using the following expresion

$$\alpha_{r,k} = \int_0^{2r-1} |\bar{K}_k(\theta)|\, d\theta$$

for all $k = 1,..., 2r-1$.

Remark 2. Following Lemma 1 and using (A.2), (A.3) and (A.4) we can show that L_k vanishes for all $p \in \mathscr{P}_{2r+1}$ as long as k is even. In this case we obtain

$$\delta_i^0 e^{(k)} = \int_a^b K_k^*(t) f^{(2r+2)}(t)\, dt$$

where

$$K_k^*(t) = \frac{1}{(2r+1)!} L_{k,x}[(x-t)_+^{2r+1}]. \tag{24}$$

In this case, if we consider $f \in C_p^{2r+2}[a,b]$ with $f^{(2r+2)}$ of bounded variation, we get

$$\|e_\Delta^{(k)}\|_\infty \leqslant \beta_{r,k}^* h^{2r+2-k}\{\|f^{(2r+2)}\|_\infty + \mathrm{Var}(f^{(2r+2)})\}$$

for all $k = 2, ..., 2r-2$, and where $\beta_{r,k}^*$ can be evaluate using (6) and (24).

In [15], Dikshit, Sharma and Tzimbalario extend the results of Lucas [10] to the case of even order periodic spline interpolation at midknots. In the same way, our results could be extended to odd order periodic spline interpolation at midknots.

APPENDIX: Properties of the Coefficients $C_{j,n}^{(k)}$

In this appendix, we recall the properties of the coefficients $C_{j,n}^{(k)}$ and extend the last one.

PROPOSITION 1.. *The coefficients $C_{j,n}^{(k)}$ have the following properties.*

(i) $$C_{j,n+1}^{(k)} = (n+1-j)\, C_{j-1,n}^{(k)} + (j+1)\, C_{j,n}^{(k)}, \qquad 0 \leqslant j \leqslant n, \quad 0 \leqslant k \leqslant n-1,$$

and

$$C_{j,k+1}^{(k)} = (-1)^{k+j}\binom{k}{j}, \qquad 0 \leqslant j \leqslant k.$$

Since $\binom{k}{j} = \binom{k-1}{j} + \binom{k-1}{j-1}$ *it follows that*

$$C_{j,k+1}^{(k)} = C_{j-1,k}^{(k-1)} - C_{j,k}^{(k-1)}.$$

(ii) $$C_{j,n}^{(k)} = (-1)^k\, C_{n-1-j,n}^{(k)}, \qquad 0 \leqslant k \leqslant n-1.$$

(iii) $$\sum_{j=0}^{n-1} C_{j,n}^{(k)} z^{j+1} = (-1)^k (1-z)^{n+1} \times \left(z\frac{d}{dz}\right)^{n-k}\left(\frac{1}{1-z}\right), \qquad 0 \leqslant k \leqslant n-1.$$

(iv) $$\sum_{j=0}^{n-1} C_{j,n}^{(k)} z^j = (z-1)^l \sum_{j=0}^{n-l-1} C_{j,n-l}^{(k-l)} z^j, \qquad l \leqslant k.$$

(v) $\sum_{j=0}^{n-1} C_{j,n}^{(0)} C_{j+k,n}^{(k)} = (-1)^k \sum_{j=0}^{n-1} C_{j,n}^{(k)} C_{j+k,n}^{(0)}, \qquad 0 \leqslant k \leqslant n-1.$

(vi) *Define the coefficients* $\alpha_{0,n} = 1$, $\alpha_{2,n}$, $\alpha_{4,n}, \ldots$ *by the equation*

$$\sinh^{n+1} x = x^{n+1} \sum_{i=0}^{\infty} \alpha_{2i,n} x^{2i}$$

and for $t = 0, 1, 2, \ldots$ *let*

$$\gamma_{t,n}^{(k)} = \frac{1}{t!} \sum_{j=0}^{n-1} C_{j,n}^{(k)} \left(j - \frac{n-1}{2}\right)^t, \qquad 0 \leqslant k \leqslant n-1. \tag{A.1}$$

Then

$$\gamma_{t,n}^{(k)} = 0 \qquad \text{if } k+t \text{ is odd or if } t < k \tag{A.2}$$

and

$$\gamma_{k+2m,n}^{(k)} = \frac{(n-k)!}{2^{2m}} \alpha_{2m,n}, \qquad 0 \leqslant m < \hat{m}, \tag{A.3}$$

$$\gamma_{k+2m,n}^{(k)} = \frac{(n-k)!}{2^{2m}} \alpha_{2m,n} + \frac{(-1)^{n+k}}{2^{2m}} \times \sum_{l=\hat{m}}^{m} \alpha_{2m-2l,n} \frac{2^{2l} B_{2l}}{2l(2l-1-n+k)!}, \qquad m \geqslant \hat{m}, \tag{A.4}$$

where $\hat{m} = [(n+2-k)/2]$, $[u]$ *denotes the integer part of* u *and* B_{2m} *denotes a Bernoulli number (see Abramowitz and Stegun* [1]).

Proof. (i) See Fyfe [9].

(ii) See Swartz [11, 12] or Albasiny and Hoskins [3].

(iii), (iv) and (v) See Albasiny and Hoskins [3].

(vi) This property has been proved for n odd by Albasiny and Hoskins [3] and by Lucas [10]. We show here how these proofs can be extended to cover the case n even.

Using the properties (ii) and setting $\hat{j} = [(n-2)/2]$, we show that

$$t!\, \gamma_{t,n}^{(k)} = (1 + (-1)^{k+t}) \sum_{j=0}^{\hat{j}} C_{j,n}^{(k)} \left(j - \frac{n-1}{2}\right)^t$$

giving $C_{j,n}^{(k)} = 0$ if $k+t$ is odd.

Taking $t=k+2m$ and using the identity $(z(d/dz))^t(z^l)=l^t z^l$, (A.1) becomes

$$\gamma_{k+2m,n}^{(k)}=\frac{1}{(k+2m)!}\left[\left(z\frac{d}{dz}\right)^{k+2m}\sum_{j=0}^{n-1} C_{j,n}^{(k)} z^{j-(n-1)/2}\right]_{z=1}.$$

Moreover, the property (iii) implies that

$$\gamma_{k+2m,n}^{(k)}=\frac{(-1)^k}{(k+2m)!}\left[\left(z\frac{d}{dz}\right)^{k+2m}\frac{(1-z)^{n+1}}{z^{(1/2)(n+1)}}\left(z\frac{d}{dz}\right)^{n-k}\left(\frac{1}{1-z}\right)\right]_{z=1}.$$

Making the substitution $z=e^{2x}$, the operator $z(d/dz)$ and $\frac{1}{2}(d/dx)$ are equivalent and $z=1$ correspond to $x=0$. Noting that $1/(1-e^{2x})=\frac{1}{2}(1-\coth x)$, we obtain

$$\gamma_{k+2m,n}^{(k)}=\frac{(-1)^{n+k}}{(k+2m)!\,2^{2m}}\left[\left(\frac{d}{dx}\right)^{k+2m}\right.$$
$$\left.\times\left\{\sinh^{n+1}x\left(\frac{d}{dx}\right)^{n-k}\coth x\right\}\right],\qquad x=0.$$

Since (see Abramowitz and Stegun [1])

$$\coth x=\sum_{l=0}^{\infty}\frac{2^{2l}}{(2l)!}B_{2l}x^{2l-1},\qquad |x|<\pi$$

it follows that

$$\left(\frac{d}{dx}\right)^{n-k}\coth x=\frac{(-1)^{n-k}(n-k)!}{x^{n-k+1}}+\sum_{l=\hat{m}}^{\infty}\frac{2^{2l}B_{2l}}{2l(2l-1-n+k)!}x^{2l-1-n+k}$$

where $\hat{m}=[(n+2-k)/2]$. Then

$$(-1)^{n+k}\sinh^{n+1}x\left(\frac{d}{dx}\right)^{n-k}\coth x=\sum_{m=0}^{\infty}\beta_{2m,n}x^{k+2m}$$

where

$$\beta_{2m,n}=(n-k)!\,\alpha_{2m,n},\qquad 0\leqslant m<\hat{m},$$
$$=(n-k)!\,\alpha_{2m,n}+(-1)^{n+k}$$
$$\times\sum_{l=\hat{m}}^{m}\alpha_{2m-2l,n}\frac{2^{2l}B_{2l}}{2l(2l-1-n+k)!},\qquad m\geqslant\hat{m}.$$

So the result follows by applying $(d/dt)^{k+2m}$ and setting $x=0$. Q.E.D.

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